# ON THE EXPECTED NUMBER OF ZEROS OF NONLINEAR EQUATIONS

GREGORIO MALAJOVICH

ABSTRACT. This paper investigates the expected number of roots of nonlinear equations. Those equations are assumed to be analytic, and to belong to certain inner product spaces. Those spaces are then endowed with the Gaussian probability distribution.

The root count on a given domain is proved to be 'additive' with respect to a product operation of functional spaces. This allows to deduce a general theorem relating the expected number of roots for unmixed and mixed systems. Examples of root counts for equations that are not polynomials nor exponential sums are given at the end.

## 1. INTRODUCTION

We consider systems of analytic equations of the form

$$f_1(\mathbf{x}) = \cdots = f_n(\mathbf{x}) = 0$$

where $\mathbf{x}$ is assumed to belong to a complex $n$-dimensional manifold $M$. Each $f_i$ belongs to a certain complex inner product space $\mathcal{F}_i$. Those will be called spaces of complex fewnomials, or *fewspaces* for short (see definition 5).

Let $n_M(\mathbf{f})$ denote the number of isolated roots of the system above. More generally, let $n_{\mathcal{K}}(\mathbf{f})$ be the number of isolated roots in a set $\mathcal{K}$. A consequence of Brouwer's degree theorem is that when $\mathcal{K}$ is open, the number $n_{\mathcal{K}}(\mathbf{f})$ is lower semi-continuous as a function of $\mathbf{f}$ (details in [22, Ch.3]).

When the $\mathcal{F}_i$ are spaces of polynomials (resp. Laurent polynomials) and $M = \mathbb{C}^n$ (resp. $M = (\mathbb{C} \setminus \{0\})^n$), the number $n_M(\mathbf{f})$ is known to be be equal to its maximum *generically*, that is for all $\mathbf{f}$ except in a codimension 1 (hence measure zero) variety. Bounds for this maximum are known, and some of them are exact.

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For instance, let $\mathcal{F}_A$ be the set of Laurent polynomials with support $A$, viz.

$$f(\mathbf{x}) = \sum_{\mathbf{a}\in A} f_{\mathbf{a}} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n},$$

where $A \subset \mathbb{Z}^n$ is assumed to be finite and $f_{\mathbf{a}} \in \mathbb{C}$. The inner product in $\mathcal{F}_A$ is arbitrary. Let $\mathcal{A}$ denote the convex hull of $A$.

**Theorem 1** (Kushnirenko [19]). *Let $f_1, \cdots, f_n \in \mathcal{F}_A$. For a generic choice of coefficients $f_{i\mathbf{a}} \in \mathbb{C}$,*

$$n_{(\mathbb{C}\setminus\{0\})^n}(\mathbf{f}) = n!\ \mathrm{Vol}(\mathcal{A}).$$

The case $n = 1$ was known to Newton, and $n = 2$ was published by Minding [31] in 1841. A system as above, where all the equations have same support $A$ is said to be *unmixed*. Otherwise, the system is said to be *mixed*. The following root count for mixed polynomial systems was published by Bernstein [4] and is known as the BKK bound (for Bernstein, Kushnirenko and Khovanskii) [5]:

**Theorem 2** (Bernstein). *Let $A_1, \ldots, A_n \subset \mathbb{Z}^n$ be finite sets. Let $\mathcal{A}_i$ be the convex hull of $A_i$. For a generic choice of coefficients $f_{i\mathbf{a}} \in \mathbb{C}$,*

$$n_{(\mathbb{C}\setminus\{0\})^n}(\mathbf{f})$$

*is $n!$ times the coefficient $V$ of $\lambda_1 \ldots \lambda_n$ in the polynomial*

$$\frac{1}{n!}\mathrm{Vol}(\lambda_1\mathcal{A}_1 + \cdots + \lambda_n\mathcal{A}_n).$$

This number $V$ is known as the *mixed volume* of the tuple of convex bodies $(\mathcal{A}_1, \ldots, \mathcal{A}_n)$.

The objective of this paper is to extend the results above to more general spaces of analytic equations. For instance, we would like to count zeros of equations such as

$$\begin{aligned} &f_{00} + f_{01}x + f_{02}x^2 + \cdots + f_{0d}x^d + \\ &\qquad + f_{10}e^x + f_{01}xe^x + f_{02}x^2e^x + \cdots + f_{0d}x^de^x = 0. \end{aligned} \tag{1}$$

It is easy to see that the number of solutions in $\mathbb{C}$ for (say) $d = 0$ is infinite. However, we can inquire about the number of solutions in a smaller set, like the disk $\mathcal{D} = \{x \in \mathbb{C} : |x| < 1\}$.

Instead of counting the *generic* number of zeros (that exists no more), we endow the space of equations with a probability measure (zero average, unit variance normal distribution) and compute the *expected* number of isolated roots.

In the example above, the expected root count is

$$\mathbb{E}(n_{\mathcal{D}}(f)) = d/2 + 0.202, 918, 921, 282 \cdots$$

(see Section 4 for the precise inner product we are using). The constant $0.202\cdots$ was obtained numerically. I would like to thank Steven Finch for pointing out an error in the 4-th decimal of a previous computation, and giving the correct decimal expansion.

This and other examples are worked out in Section 6

It turns out that complex fewnomial spaces are reproducing kernel spaces. A meaningful multiplication operation between reproducing kernel spaces was studied by Aronszajn [2] (see Section 4). We denote the product space of $\mathcal{F}$ and $\mathcal{G}$ by $\mathcal{F}\mathcal{G}$, and the $\lambda$-th power of $\mathcal{F}$ by $\mathcal{F}^\lambda$. The main result in this paper is an analogous to Bernstein's theorem. However, there is no more an interpretation of the number of roots in terms of a volume of a convex body (Minding and Kushnirenko) or in terms of mixed volume. But the relation between root counts in mixed and unmixed systems is preserved.

**Theorem 3.** *Let $\mathcal{F}_1, \dots, \mathcal{F}_n$ be finite dimensional fewspaces of functions of $M$, endowed with the zero average unit variance normal probability distribution. Let $\mathcal{K} \subseteq M$ be measurable. Then,*

$$\mathbb{E}_{f_1 \in \mathcal{F}_1, \dots, f_n \in \mathcal{F}_n}(n_{\mathcal{K}}(\mathbf{f}))$$

*is the coefficient of $\lambda_1 \lambda_2 \cdots \lambda_n$ in the $n$-th degree homogeneous polynomial*

$$\frac{1}{n!}\mathbb{E}_{g_1, \dots, g_n \in \mathcal{F}_1^{\lambda_1}\mathcal{F}_2^{\lambda_2}\cdots\mathcal{F}_n^{\lambda_n}}(n_{\mathcal{K}}(\mathbf{g}))$$

*where zero average and unit variance normal probability distribution is assumed in each $\mathcal{F}_1^{\lambda_1}\mathcal{F}_2^{\lambda_2}\cdots\mathcal{F}_n^{\lambda_n}$.*

In the setting of Bernstein's theorem, one may identify $\mathcal{F}_{\lambda_1 A_1 + \cdots + \lambda_n A_n}$ to $\mathcal{F}_{A_1}^{\lambda_1}\mathcal{F}_{A_2}^{\lambda_2}\cdots\mathcal{F}_{A_n}^{\lambda_n}$. With this identification, Bernstein's theorem follows immediately from Kushnirenko's theorem and Theorem 3.

The basic idea for the proof of Theorem 3 is:

**Lemma 4.** *Let $\mathcal{E}, \mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n$ be finite dimensional fewspaces and let $\lambda \geq 0$ be integer. Then,*

$$\begin{aligned}\mathbb{E}_{f_1 \in \mathcal{E}^\lambda \mathcal{F}_1, f_2 \in \mathcal{F}_2, \dots, f_n \in \mathcal{F}_n}(n_{\mathcal{K}}(\mathbf{f})) &= \mathbb{E}_{f_1 \in \mathcal{E}, f_2 \in \mathcal{F}_2, \dots, f_n \in \mathcal{F}_n}(n_{\mathcal{K}}(\mathbf{f})) + \\ &\quad + \lambda \mathbb{E}_{f_1 \in \mathcal{F}_1, f_2 \in \mathcal{F}_2, \dots, f_n \in \mathcal{F}_n}(n_{\mathcal{K}}(\mathbf{f})).\end{aligned}$$

*Above, all fewspaces are assumed with the zero average, unit variance normal probability distribution.*

## 2. Related work

Random polynomial systems constitute a classical subject of studies, and received a lot of attention lately (See for instance the book by Azaïs and Wschebor [3] and references). Part of the interest comes from the study of algorithms for solving polynomial systems such as in [26–30]. The running time of algorithms can be estimated in terms of certain invariants, such as the number of real or complex zeros, and the condition number. While the number of real zeros of real polynomial systems and the condition number depend on the input system, it is possible to obtain probabilistic complexity estimates by endowing the space of polynomials with a probability distribution, and then treating those quantities as random variables. For the full picture, see the book [6] and two forthcoming books [8, 22]. Recent papers on the subject include [1, 10–12]. The extension of this theory to systems of sparse polynomial systems started with [23, 24] (see below) and is still a research subject (see [22]).

Another source of interest comes from classical asymptotic estimates such as in Littlewood-Offord [20, 21] and Kac [14, 15].

Asymptotic formulas for the number of roots of sparse polynomial systems can be obtained by scaling the supports. For instance, one looks at systems of Laurent polynomials such as

$$f_i(\mathbf{x}) = \sum_{\mathbf{a} \in A_i} f_{i\mathbf{a}} \mathbf{x}^{t\mathbf{a}}$$

where $t$ is a scaling parameter. A random variable of interest in the zero-dimensional case is $t^{-n} n_M(\mathbf{f})$. In [25], Shiffman and Zelditch gave asymptotic formulas for the root density in terms of the mixed volume form.

Kazarnovskii [17] obtained more general formulas. He considered fewnomials that are (after multiplying variables by $\sqrt{-1}$) Fourier transforms of distributions supported by real compact sets. For instance, (1) is the Fourier transform of a distribution with support $\{0, 1\}$, namely

$$\sum_{\substack{i=0,1 \\ j=0,\dots,d}} f_{ij} \frac{(-1)^j}{j!} \delta_i^{(j)}(y).$$

The convex bodies that appear in the Kushnirenko and Bernstein theorems are replaced by the convex hull of the support of the distributions. In this sense, he generalized Bernstein's theorem to non-polynomials and non-exponential-sums. However, his bounds for (say)

(1) do not take into account different values of $d$. That is why those bounds must be asymptotic.

## 3. Spaces of complex fewnomials

Let $M$ be an $n$-dimensional complex manifold. In this section we review part of the theory of spaces of complex fewnomials in $M$. This theory is developed in more details in [22]. Canonical references for analytic functions of several variables and for reproducing kernel spaces are, respectively, [18] and [2].

**Definition 5.** A *complex fewnomial space* (or *fewspace* for short) of functions over a complex manifold $M$ is a Hilbert space of holomorphic functions from $M$ to $\mathbb{C}$ such that the following holds. Let $V : M \to \mathcal{F}^*$ denote the *evaluation form* $V(\mathbf{x}) : f \mapsto f(\mathbf{x})$. For any $\mathbf{x} \in M$,

(1) $V(\mathbf{x})$ is a continuous linear form.
(2) $V(\mathbf{x})$ is not the zero form.

In addition, we say that the fewspace is *non-degenerate* if and only if, for any $\mathbf{x} \in M$,

3. $P_{V(\mathbf{x})} DV(\mathbf{x})$ has full rank,

where $P_W$ denotes the orthogonal projection onto $W^{\perp}$. (The derivative is with respect to $\mathbf{x}$). In particular, a non-degenerate fewspace has dimension $\geq n+1$.

**Example 6.** Let $M$ be an open connected subset of $\mathbb{C}^n$. *Bergman space* $\mathcal{A}(M)$ is the space of holomorphic functions defined in $M$ with finite $\mathcal{L}^2$ norm. The inner product is the $\mathcal{L}^2$ inner product. When $M$ is bounded, $\mathcal{A}(M)$ contains constant and linear functions, hence it is a non-degenerate fewspace.

*Remark* 7. Condition 1 holds trivially for any finite dimensional fewnomial space, and less trivially for subspaces of Bergman space.

To each fewspace $\mathcal{F}$ we associate two objects: The *reproducing kernel* $K(\mathbf{x}, \mathbf{y}) = K_{\mathcal{F}}(\mathbf{x}, \mathbf{y})$ and a possibly degenerate Kähler form $\omega = \omega_{\mathcal{F}}$ on $M$.

Item (1) in the definition makes $V(\mathbf{x})$ an element of the dual space $\mathcal{F}^*$ of $\mathcal{F}$ (more precisely, the space of continuous functionals $\mathcal{F} \to \mathbb{C}$).

Riesz-Fréchet representation Theorem (e.g. [7] Th.V.5 p.81) allows to identify $\mathcal{F}$ and $\mathcal{F}^*$, whence the Kernel $K(\mathbf{x}, \mathbf{y}) = \overline{(V(\mathbf{x})^*)(\mathbf{y})}$.

For fixed $\mathbf{x}$, $K(\mathbf{x}, \mathbf{y}) \in \mathcal{F}$ as a function of $\bar{\mathbf{y}}$.

By construction, for $f \in \mathcal{F}$,

$$f(\mathbf{y}) = \langle f(\cdot), K(\cdot, \mathbf{y}) \rangle.$$

There are two consequences. First of all,

$$K(\mathbf{y}, \mathbf{x}) = \langle K(\cdot, \mathbf{x}), K(\cdot, \mathbf{y}) \rangle = \overline{\langle K(\cdot, \mathbf{y}), K(\cdot, \mathbf{x}) \rangle} = \overline{K(\mathbf{x}, \mathbf{y})}$$

and in particular, for any fixed $\mathbf{y}$, $\mathbf{x} \mapsto K(\mathbf{x}, \mathbf{y})$ is also an element of $\mathcal{F}$. Thus, $K(\mathbf{x}, \mathbf{y})$ is analytic in $\mathbf{x}$ and in $\bar{\mathbf{y}}$. Moreover, $\|K(\mathbf{x}, \cdot)\|^2 = K(\mathbf{x}, \mathbf{x})$.

Secondly, $Df(\mathbf{y})\dot{\mathbf{y}} = \langle f(\cdot), D_{\bar{\mathbf{y}}} K(\cdot, \mathbf{y}) \bar{\dot{\mathbf{y}}} \rangle$ and the same holds for higher derivatives.

Because of Definition 5(2), $K(\cdot, y) \neq 0$. Thus, $y \mapsto K(\cdot, y)$ induces a map from $M$ to $\mathbb{P}(\mathcal{F})$. The differential form $\omega$ is defined as the pull-back of the Fubini-Study form

$$\omega_f = \frac{\sqrt{-1}}{2} \partial \bar{\partial} \log \|f\|^2$$

of $\mathbb{P}(\mathcal{F})$ by $y \mapsto K(\cdot, y)$.

Namely,

$$(\omega_{\mathcal{F}})_{\mathbf{x}} = \omega_{\mathbf{x}} = \frac{\sqrt{-1}}{2} \partial \bar{\partial} \log K(\mathbf{x}, \mathbf{x}). \tag{2}$$

When the form $\omega$ is non-degenerate for all $x \in M$, it induces a Hermitian structure on $M$. This happens if and only if the fewspace is a non-degenerate fewspace.

*Remark* 8. If $\mathcal{F} = \mathcal{A}(M)$ is the Bergman space, the kernel obtained above is known as the *Bergman Kernel* and the metric induced by $\omega$ as the *Bergman metric*.

*Remark* 9. If $\phi_i(x)$ denotes an orthonormal basis of $\mathcal{F}$ (finite or infinite), then the kernel can be written as

$$K(\mathbf{x}, \mathbf{y}) = \sum \phi_i(\mathbf{x}) \overline{\phi_i(\mathbf{y})}.$$

Let $n_{\mathcal{K}}(f)$ be the number of isolated zeros of $f$ that belong to a measurable set $\mathcal{K}$. The following result is well-known. It appears in [16, Prop.3] and [13, Prop-Def.1.6A]. It is a consequence of *Crofton's formula*, also knwon as *Rice formula* or *coarea formula*.

**Theorem 10** (Root density)**.** *Let $\mathcal{K}$ be a locally measurable set of an $n$-dimensional manifold $M$. Let $\mathcal{F}_1, \dots, \mathcal{F}_n$ be fewspaces. Let $\omega_1, \dots, \omega_n$ be the induced symplectic forms on $M$. Assume that $\mathbf{f} = f_1, \dots, f_n$ is a zero average, unit variance variable in $\mathcal{F} = \mathcal{F}_1 \times \cdots \times \mathcal{F}_n$. Then,*

$$\mathbb{E}(n_{\mathcal{K}}(\mathbf{f})) = \frac{1}{\pi^n} \int_{\mathcal{K}} \omega_1 \wedge \cdots \wedge \omega_n.$$

As the formulation in terms of reproducing kernel spaces is not standard, we sketch the proof below (more details are available in [22, Th.5.11]).

*Proof.* First of all, let $\mathcal{V} = \{(\mathbf{f}, \mathbf{x}) \in \mathcal{F} \times \mathcal{K} : \mathbf{f}(\mathbf{x}) = 0\}$ be the incidence variety, and $\pi_1 : \mathcal{V} \to \mathcal{F}$, $\pi_2 : \mathcal{V} \to \mathcal{K}$ be the canonical projections.

In a neighborhood of each regular point $(\mathbf{f}_0, \mathbf{x}_0)$ of $\mathcal{V}$, it is possible to parametrize $\mathcal{V}$ by an implicit function $(G(\mathbf{f}), \mathbf{f})$ with $G(\mathbf{f}_0) = \mathbf{x}_0$ and

$$DG(\mathbf{x}_0) = -D\mathbf{f}(\mathbf{x}_0)^{-1} \left(K_1(\mathbf{x}_0, \cdot)^* \otimes \cdots \otimes K_n(\mathbf{x}_0, \cdot)^*\right)$$

where $K_i$ is the reproducing kernel of $\mathcal{F}_i$.

Let $\mathcal{F}_\mathbf{x}$ denote the product $K_1(\mathbf{x}, \cdot)^\perp \times \cdots \times K_n(\mathbf{x}, \cdot)^\perp$.

The coarea formula is now

$$\begin{aligned} \mathbb{E}(n_\mathcal{K}(\mathbf{f})) &= \frac{1}{(2\pi)^{\dim \mathcal{F}}} \int_\mathcal{F} \# \pi_2 \circ \pi_1^{-1}(\{) e^{-\|\mathbf{f}\|^2/2} \, \mathrm{d}V_\mathcal{F}(\mathbf{f}) \\ &= \frac{1}{(2\pi)^{\dim \mathcal{F}}} \int_\mathcal{K} dV_M(\mathbf{x}) \int_{\mathcal{F}_\mathbf{x}} NJ(\mathbf{f}, \mathbf{x})^{-2} e^{-\|\mathbf{f}\|^2/2} \, \mathrm{d}V_{\mathcal{F}_\mathbf{x}}(\mathbf{f}) \end{aligned}$$

with $NJ = \det\left(DG(\mathbf{x}) DG(\mathbf{x})^*\right)^{1/2} = |\det D\mathbf{f}(\mathbf{x})|^{-1} \prod (K_i(\mathbf{x}, \mathbf{x}))^{1/2}$.

The reader may check that

$$\det |D\mathbf{f}(\mathbf{x})|^2 \mathrm{d}V = \bigwedge_{i=1^n} \sum_{j,k=1}^n \frac{\partial}{\partial x_j} f_i(\mathbf{x}) \overline{\frac{\partial}{\partial x_k} f_i(\mathbf{x})} \, \frac{\sqrt{-1}}{2} \mathrm{d}x_j \wedge d\bar{x}_k$$

(otherwise, this is Lemma 4.3 in [22]). At this point,

$$\mathbb{E}(n_\mathcal{K}(\mathbf{f})) = \frac{1}{(2\pi)^n} \int_\mathcal{K} dV_M(\mathbf{x}) \bigwedge_{i=1}^n \Omega_i$$

with

$$\Omega_i = \int_{K_i(\mathbf{x}, \cdot)^\perp} \frac{\frac{\partial}{\partial x_j} f_i(\mathbf{x}) \overline{\frac{\partial}{\partial x_k} f_i(\mathbf{x})} \, \frac{\sqrt{-1}}{2} \mathrm{d}x_j \wedge d\bar{x}_k}{K_i(\mathbf{x}, \mathbf{x})} \frac{e^{-\|f_i\|^2/2}}{(2\pi)^{\dim \mathcal{F}_i - 1}} \, \mathrm{d}V_{K_i(\mathbf{x}, \cdot)^\perp}(f_i).$$

The result below implies that $\Omega_i = 2\omega_i$, concluding the proof of the density theorem. □

**Proposition 11.** *Let* $\langle \mathbf{u}, \mathbf{w} \rangle_{i,\mathbf{x}} = \omega_{i,\mathbf{x}}(\mathbf{u}, J\mathbf{w})$ *be the (possibly degenerate) Hermitian product associated to* $\omega_i$*. Then,*

$$\langle \mathbf{u}, \mathbf{w} \rangle_{i,\mathbf{x}} = \frac{1}{2} \int_{K_i(\mathbf{x}, \cdot)^\perp} \frac{(Df_i(\mathbf{x})\mathbf{u}) \overline{Df_i(\mathbf{x})\mathbf{w}}}{K_i(\mathbf{x}, \mathbf{x})} \frac{e^{-\|f_i\|^2}}{(2\pi)^{\dim \mathcal{F}_i - 1}} \, \mathrm{d}V_{K_i(\mathbf{x}, \cdot)^\perp}(f_i).$$

*Proof of Proposition 11.* Let

$$P_\mathbf{x} = I - \frac{K_i(\mathbf{x}, \cdot) K_i(\mathbf{x}, \cdot)^*}{K_i(\mathbf{x}, \mathbf{x})}$$

be the orthogonal projection. Since the inner product $\langle \cdot, \cdot \rangle_i$ is the pull-back of Fubini-Study by $\mathbf{x} \mapsto K_i(\mathbf{x}, \cdot)$, we can write the left-hand-side as:

$$\langle \mathbf{u}, \mathbf{w} \rangle_{i,\mathbf{x}} \quad = \quad \frac{\langle P_{\mathbf{x}} DK_i(\mathbf{x}, \cdot)\mathbf{u}, P_{\mathbf{x}} DK_i(\mathbf{x}, \cdot)\mathbf{w} \rangle}{K_i(\mathbf{x}, \mathbf{x})}$$

For the right-hand-side, note that

$$Df_i(\mathbf{x})\mathbf{u} = \langle f_i(\cdot), DK_i(\cdot, \mathbf{x})\mathbf{u} \rangle = \langle f_i(\cdot), P_{\mathbf{x}} DK_i(\cdot, \mathbf{x})\mathbf{u} \rangle.$$

Let $\mathbf{U} = \frac{1}{\|K_i(\cdot,\mathbf{x})\|} P_{\mathbf{x}} DK_i(\cdot, \mathbf{x})\mathbf{u}$ and $\mathbf{W} = \frac{1}{\|K(\cdot,\mathbf{x})\|} P_{\mathbf{x}} DK(\cdot, \mathbf{x})\mathbf{w}$. Both $\mathbf{U}$ and $\mathbf{W}$ belong to $\mathcal{F}_{\mathbf{x}}$. The right-hand-side is

$$\begin{aligned}
\frac{1}{2} \int_{K_i(\mathbf{x},\cdot)^\perp} \frac{(Df_i(\mathbf{x})\mathbf{u})\overline{Df_i(\mathbf{x})\mathbf{w}}}{\|K_i(\mathbf{x}, \mathbf{x})\|^2} \frac{e^{-\|f_i\|^2}}{(2\pi)^{\dim \mathcal{F}_i - 1}} \, \mathrm{d}V_{K_i(\mathbf{x},\cdot)^\perp}(f_i) &= \frac{1}{2} \int_{K_i(\mathbf{x},\cdot)^\perp} \langle f_i, \mathbf{U} \rangle \overline{\langle f_i, \mathbf{W} \rangle} \frac{e^{-\|f_i\|^2}}{(2\pi)^{\dim \mathcal{F}_i}} \\
&= \frac{1}{2} \langle \mathbf{U}, \mathbf{W} \rangle \int_{\mathbb{C}} \frac{1}{2\pi} |z|^2 e^{-|z|^2/2} \, \mathrm{d}z \\
&= \langle \mathbf{U}, \mathbf{W} \rangle
\end{aligned}$$

which is equal to the left-hand-side. $\square$

## 4. Product spaces

Let $\mathbf{E}$ and $\mathbf{F}$ be complex inner product spaces. If $e \in \mathbf{E}$ and $f \in \mathbf{F}$, we denote by $e \otimes f$ the class of equivalence of pairs $(e, f)$ under $(\lambda e, f) \sim (e, \lambda f)$. The *tensor product* of $\mathbf{E}$ and $\mathbf{F}$ is the space of all linear combinations of elements of the form $e \otimes f$. In the case $\mathbf{E}$ and $\mathbf{F}$ are finite dimensional, $\mathbf{E} \otimes \mathbf{F}$ can be assimilated to the space of bilinear maps $\mathbf{E} \times \mathbf{F} \to \mathbb{C}$.

The canonical inner product for the tensor product of two spaces is given by

$$\langle e_1 \otimes f_1, e_2 \otimes f_2 \rangle_{\mathbf{E} \otimes \mathbf{F}} = \langle e_1, e_2 \rangle_{\mathbf{E}} \langle f_1, f_2 \rangle_{\mathbf{F}} .$$

Now, let $\mathcal{E}$ and $\mathcal{F}$ be fewnomial spaces on some complex manifold $M$. Then, $\mathcal{E} \otimes \mathcal{F}$ is a fewnomial space on the product $M \times M$, where we interpret $(e \otimes f)(x_1, x_2) = e(x_1)f(x_2)$. A classical fact on reproducing kernel spaces allows to recover the kernel of the tensor product:

**Theorem 12** (Aronszajn)**.** *The direct (=tensor) product $\mathcal{E} \otimes \mathcal{F}$ posseses the reproducing kernel*

$$K_{\mathcal{E} \otimes \mathcal{F}}((x_1, x_2), (y_1, y_2)) = K_{\mathcal{E}}(x_1, y_1) \, K_{\mathcal{F}}(x_2, y_2)$$

.

This is [2, Theorem I p.361]. Theorem II ibid gives us a convenient notion of 'product' for reproducing kernel spaces with same domain:

**Theorem 13** (Aronszajn)**.** *The kernel $K_{\mathcal{G}}(x,y) = K_{\mathcal{E}}(x,y)K_{\mathcal{F}}(x,y)$ is the reproducing kernel of the class $\mathcal{G}$ of restrictions of all functions of the direct (=tensor) product $E \otimes F$ to the diagonal set $M_1 = \{(x,x) : x \in M\} \simeq M$. For any such restriction, $\|g\| = \min \|g'\|_{E\otimes F}$, the restriction of which to the diagonal set $M_1$ is $g$.*

If $\mathcal{E}$ and $\mathcal{F}$ are spaces of fewnomials on $M$, we denote by $\mathcal{E}\mathcal{F}$ the class $\mathcal{G}$ described above. As an inner product space, $\mathcal{G}$ is just the orthogonal complement of the kernel of the restriction operator

$$\begin{array}{rccl} \Delta: & \mathcal{E}\otimes\mathcal{F} & \rightarrow & \mathcal{O}(M), \\ & g' & \mapsto & g = g'_{|M'} \end{array}$$

The inner product of $\mathcal{G}$ is by definition the inner product of $\mathcal{E}\otimes\mathcal{F}$ restricted to $(\ker\Delta)^{\perp}$.

Given orthormal bases $(e_a)_{a\in A}$ and $(f_b)_{b\in B}$ of $\mathcal{E}$ and $\mathcal{F}$, we can produce an orthonormal basis of $\mathcal{G}$ as follows.

First, we notice that $(e_a \otimes f_b)_{(a,b)\in A\times B}$ is an orthonormal basis of $\mathcal{E}\otimes\mathcal{F}$.

Let $\sim$ be the equivalence relation of $A \times B$ given by

$$(a,b) \sim (a',b') \text{ if and only if } e_a(x)f_b(x) \equiv \pm e_{a'}(x)f_{b'}(x).$$

Let $C = \frac{A\times B}{\sim}$. For each equivalence class $c \in C$ (as a subset of $A \times B$), choose a pair $(a,b) \in c$ and set

$$g_c = \sqrt{\#c}\; e_a f_b.$$

**Lemma 14.** *$(g_c)_{c\in C}$ is an orthonormal basis of $\mathcal{G}$.*

*Proof.* Define

$$G_c = \frac{1}{\sqrt{\#c}} \sum_{(a,b)\in c} \sigma_{ab} e_a \otimes f_b$$

where $\sigma_{ab} = \pm 1$ with sign choosed so that

$$\sqrt{\#c}\; \sigma_{ab} e_a f_b = g_c.$$

The $(G_c)_{c\in C}$ are linearly independent, hence $(g_c)_{c\in C}$ is a linearly independent set.

In order to show that the span of $(g_c)_{c\in C}$ is $\ker\Delta^{\perp}$, let

$$H = \sum_{(a,b)\in A\times B} H_{ab} e_a \otimes f_b \in \ker\Delta.$$

This implies that

$$\sum_{c\in C} \frac{1}{\sqrt{\#c}} \sum_{(a,b)\in c} H_{ab}\sigma_{ab} g_c = 0$$

and hence for all $c$,

$$\langle H, G_c \rangle = \frac{1}{\# c} \sum H_{ab} \sigma_{ab} = 0.$$

Reciprocally, let $G \in \ker \Delta^{\perp}$. Write

$$G = \sum_{(a,b) \in A \times B} e_a \otimes f_b.$$

By hypothesis, $G \perp \sigma_{ab} e_a \otimes f_b - \sigma_{a'b'} e_{a'} \otimes f_{b'}$ whenever $(a,b) \sim (a',b')$. Therefore, $G$ is a linear combination of the $G_c$.

Finally, it is easy to check that

$$\langle G_c, G_{c'} \rangle = \begin{cases} 0 & \text{when } c \neq c' \\ 1 & \text{when } c = c'. \end{cases}$$

□

**Lemma 15.** *Let $M$ be fixed. The product of fewspaces of $M$ is associative and comutative. If one introduces the 'constant' fewspace $\mathcal{I} = \{1\}$, then fewspaces on $M$ are a semigroup.*

**Example 16.** Let $M = \mathbb{C}^n$ and let $\mathcal{P}_1$ be the space of affine functions in $n$ variables. To make it an inner product space, we assume that $(1, x_1, x_2, \cdots, x_n)$ is an orthonormal basis. We define inductively $\mathcal{P}_{d+1} = \mathcal{P}_d \mathcal{P}_1$. Here is the orthonormal basis of $\mathcal{P}_d$:

$$\left( \sqrt{\binom{d}{a_0 a_1, \dots, a_n}} x_1^{a_1} x_2^{a_2} \cdots x_n^{a^n} \right)_{\substack{a_0, \dots, a_n \geq 0 \\ \sum_{0 \leq j \leq n} a_j = d}}$$

Above, the multinomial coefficient

$$\binom{d}{a_0, a_1, \dots, a_n} = \frac{d!}{a_0! \cdots a_n!}$$

is the number of ways to distribute $d = a_0 + \cdots + a_n$ balls into $n + 1$ numbered buckets of size $a_0, \dots, a_n$. It is also the coefficient of $x_1^{a_1} x_2^{a_2} \cdots x_n^{a^n}$ in $(1 + x_1 + \cdots + x_n)^d$. This corresponds to the unitarily invariant inner product defined by Weyl [33], also known as Bombieri's.

The reproducing kernel of $\mathcal{P}_d$ is easily seen to be

$$K_d(\mathbf{x}, \mathbf{y}) = (1 + x_1 \bar{y}_1 + \cdots + x_n \bar{y}_n)^d.$$

With the same formalism, we can also retrieve the multi-unitarily invariant inner product for the space of roots of multihomogeneous polynomial systems introduced by Rojas [32].

**Example 17.** Let $A \subseteq (\mathbb{Z})^n$ be finite, and $M = (\mathbb{C}_{\neq 0})^n$. Let $\mathcal{F}_A$ be the space of Laurent polynomials of the form

$$f(\mathbf{x}) = \sum_{\mathbf{a} \in A} \mathbf{x}^{\mathbf{a}}.$$

Assume arbitrary weights $c_{\mathbf{a}} > 0$ so that $\|x_{\mathbf{a}}\|_A^2 = c_{\mathbf{a}}$. Then,

$$K_A(\mathbf{x}, \mathbf{y}) = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} \bar{\mathbf{y}}^{\mathbf{a}}.$$

The $\lambda$-th power $\mathcal{F}_A^\lambda$ of $\mathcal{F}_A$ is precisely $\mathcal{F}_B$ with $B = \lambda \operatorname{Conv}(A) \cap \mathbb{Z}^n$ and weights

$$c_{\mathbf{b}} = \sum_{\mathbf{a}_1 + \cdots + \mathbf{a}_\lambda = \mathbf{b}} c_{\mathbf{a}_1} c_{\mathbf{a}_2} \cdots c_{\mathbf{a}_\lambda}.$$

An orthonormal basis is $(c_{\mathbf{b}}^{-1/2} x^{\mathbf{b}})$.

## 5. Proof of the main results

*Proof of Lemma 4.* Let $\mathcal{E}$ and $\mathcal{F}_1$ be fewspaces on a complex manifold $M$, and let $\mathcal{G} = \mathcal{E}^\lambda \mathcal{F}$. By Theorem 13,

$$K_{\mathcal{G}}(\mathbf{x}, \mathbf{y}) = K_{\mathcal{E}}(\mathbf{x}, \mathbf{y})^\lambda K_{\mathcal{F}}(\mathbf{x}, \mathbf{y}).$$

By (2), we deduce that

$$\omega_{\mathcal{G}} = \lambda \omega_{\mathcal{E}} + \omega_{\mathcal{F}}.$$

Now, we just insert the formula above in Theorem 10. □

*Proof of Theorem 3.* Let $\mathcal{G} = \mathcal{F}_1^{\lambda_1} \mathcal{F}_2^{\lambda_2} \cdots \mathcal{F}_n^{\lambda_n}$. Let $\omega_i$ be the Kähler form associated to the space $\mathcal{F}_i$. The form associated to $\mathcal{G}$ is

$$\omega_{\mathcal{G}} = \lambda_1 \omega_{\mathcal{F}_1} + \cdots + \lambda_n \omega_{\mathcal{F}_n}$$

Because the $\omega_i$ are 2-forms, they commute. Theorem 10 implies that

$$\begin{aligned}
&\mathbb{E}_{g_1, \dots, g_n \in \mathcal{F}_1^{\lambda_1} \mathcal{F}_2^{\lambda_2} \cdots \mathcal{F}_n^{\lambda_n}} (n_M(\mathbf{g})) = \\
&\quad = \frac{1}{\pi^n} \int_M \omega_{\mathcal{G}} \wedge \cdots \wedge \omega_{\mathcal{G}} \\
&\quad = \sum_{i_1, \dots, i_n \in \{1, \dots, n\}} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_n} \frac{1}{\pi^n} \int_M \omega_{i_1} \wedge \cdots \wedge \omega_{i_n} \\
&\quad = \sum_{i_1, \dots, i_n \in \{1, \dots, n\}} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_n} \mathbb{E}_{f_1 \in \mathcal{F}_{i_1}, \dots, f_n \in \mathcal{F}_{i_n}} (n_M(\mathbf{f})).
\end{aligned}$$

In the last line above, the monomial $\lambda_1 \lambda_2 \cdots \lambda_n$ appears $n!$ times. Its coefficient is therefore

$$n!\ \mathbb{E}_{f_1 \in \mathcal{F}_{i_1}, \dots, f_n \in \mathcal{F}_{i_n}} (n_M(\mathbf{f})).$$

□

## 6. Explicit calculation of the number of zeros

6.1. **The example in the introduction.** We start by the bound on the expected number of roots of (1) in the introduction. Let $\mathcal{E}$ denote the fewspace of functions on the disk $\mathcal{D} = \{z \in \mathbb{C} : |z| < 1\}$ spanned by 1 and $e^z$. We assume that 1 and $e^z$ form an orthonormal basis. Then

$$K_{\mathcal{E}}(x, y) = 1 + e^{x+\bar{y}}.$$

An easy computation is now

$$\omega_{\mathcal{E}} = \frac{\sqrt{-1}}{2} \partial\bar{\partial} \log K_{\mathcal{E}}(z, z) = \frac{e^{2\mathrm{Re}(z)}}{(1 + e^{2\mathrm{Re}(z)})^2}$$

The following numerical approximation was obtained by Steven Finch using Mathematica. It was independently checked by this author using long double IEEE arithmetic.

$$\mathbb{E}_{f\in\mathcal{E}}(n_f(\mathcal{D})) = \pi^{-1} \int_{\mathcal{D}} \omega = 0.202, 918, 921, 282 \cdots .$$

It is obvious that $E_{f\in\mathcal{P}_d}(n_{\mathcal{D}}(f)) = \pi^{-1} \int_{\mathcal{D}} \omega_{\mathcal{P}_d} = d/2$. Hence,

$$\mathbb{E}_{f\in\mathcal{E}\mathcal{P}_d}(n_f(\mathcal{D})) = \pi^{-1} \int_{\mathcal{D}} \omega + \omega_{\mathcal{P}_d} = d/2 + 0.202, 918, 921, 282 \cdots .$$

6.2. **An $n$-dimensional example.** We consider now systems where each equation is of the form

$$\sum f_{\mathbf{a},\mathbf{b}} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} e^{b_1 x_1 + \cdots + b_n x_n}$$

and the sum is taken for all $0 \le a_i \le d$ and $b_i = 0, 1$. The corresponding domain will be the polydisc $\mathcal{D}^n$.

The fewnomial space is

$$\mathcal{G} = (\mathcal{E}\mathcal{P}_d) \otimes (\mathcal{E}\mathcal{P}_d) \otimes \cdots \otimes (\mathcal{E}\mathcal{P}_d).$$

Let $\omega = g(z)\ \frac{\sqrt{-1}}{2} \mathrm{d}z \wedge \mathrm{d}\bar{z}$ be the Kähler form corresponding to $(\mathcal{E}\mathcal{P}_d)$. Then from Th.12 and (2), we deduce that

$$\omega_{\mathcal{G}} = \sum_{i=1}^{n} g(z_i)\ \frac{\sqrt{-1}}{2} \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i.$$

Hence,

$$\begin{aligned}
\mathbb{E}_{f_1,\ldots,f_n\in\mathcal{G}}(n_{\mathbf{f}}(\mathcal{D}^n)) &= \pi^{-n} \int_{\mathcal{D}^n} \omega_{\mathcal{G}}^{\wedge n} \\
&= n!(d/2 + 0.202, 918, 921, 282 \cdots)^n.
\end{aligned}$$

6.3. **An unmixed example.** We consider now the case where the first equation belongs to $\mathcal{G} = (\mathcal{E}\mathcal{P}_{d_1})^{\otimes n}$ as above, but the other equations polynomials of degree $d_2, \cdots, d_n$ in each variable (they belong to $\mathcal{P}_{d_j}^{\otimes n}$.

Then, let $\mathcal{H} = \mathcal{G}^{\lambda_1}\mathcal{P}_{d_2}^{\lambda_2} \cdots \mathcal{P}_{d_n}^{\lambda_n}$. Note that

$$\mathcal{H} = \mathcal{E}^{\lambda_1}\mathcal{P}_1^{\lambda_1 d_1 + \cdots + \lambda_n d_n}$$

From the previous example,

$$\frac{1}{n!}\mathbb{E}_{f_1,\dots,f_n\in\mathcal{H}}(n_{\mathbf{f}}(\mathcal{D}^n)) = \\ = \left(\frac{\lambda_1 d_1 + \cdots + \lambda_n d_n}{2} + \lambda_1 0.202,918,921,282\cdots\right)^n$$

The coefficient in $\lambda_1\lambda_2\cdots\lambda_n$ is

$$n!\frac{d_1 d_2 \cdots d_n}{2^n} + n - 1!\frac{d_2 \cdots d_n}{2^{n-1}} 0.202,918,921,282\cdots$$

By Theorem 3:

$$\mathbb{E}_{f_1\in\mathcal{G}, f_2\in\mathcal{P}_2,\cdots,f_n\in\mathcal{P}_n} n_{\mathcal{D}}(\mathbf{f}) = \\ = n!\frac{d_1 d_2 \cdots d_n}{2^n} + n - 1!\frac{d_2 \cdots d_n}{2^{n-1}} 0.202,918,921,282\cdots.$$

## References


[1] Diego Armentano and Jean-Pierre Dedieu, *A note about the average number of real roots of a Bernstein polynomial system*, J. Complexity **25** (2009), no. 4, 339–342, DOI 10.1016/j.jco.2009.03.001.

[2] N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68** (1950), 337–404.

[3] Jean-Marc Azaïs and Mario Wschebor, *Level sets and extrema of random processes and fields*, John Wiley & Sons Inc., Hoboken, NJ, 2009.

[4] D. N. Bernstein, *The number of roots of a system of equations*, Funkcional. Anal. i Priložen. **9** (1975), no. 3, 1–4 (Russian).

[5] D. N. Bernstein, A. G. Kušnirenko, and A. G. Hovanskiĭ, *Newton polyhedra*, Uspehi Mat. Nauk **31** (1976), no. 3(189), 201–202 (Russian).

[6] Lenore Blum, Felipe Cucker, Michael Shub, and Steve Smale, *Complexity and real computation*, Springer-Verlag, New York, 1998. With a foreword by Richard M. Karp.

[7] Haïm Brezis, *Analyse fonctionnelle*, Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree], Masson, Paris, 1983 (French). Théorie et applications. [Theory and applications].

[8] Peter Bürgisser and Felipe Cucker, *Conditionning*. In preparation.

[9] Felipe Cucker, Teresa Krick, Gregorio Malajovich, and Mario Wschebor, *A numerical algorithm for zero counting. I. Complexity and accuracy*, J. Complexity **24** (2008), no. 5-6, 582–605, DOI 10.1016/j.jco.2008.03.001.

[10] ———, *A numerical algorithm for zero counting. II. Distance to ill-posedness and smoothed analysis*, J. Fixed Point Theory Appl. **6** (2009), no. 2, 285–294, DOI 10.1007/s11784-009-0127-4.

[11] Felipe Cucker, Teresa Krick, Gregorio Malajovich, and Mario Wschebor, *A numerical algorithm for zero counting III: Randomization and Condition*, 2011. To appear.http://arxiv.org/abs/1007.1597.

[12] Jean-Pierre Dedieu and Gregorio Malajovich, *On the number of minima of a random polynomial*, J. Complexity **24** (2008), no. 2, 89–108, DOI 10.1016/j.jco.2007.09.003.

[13] M. Gromov, *Convex sets and Kähler manifolds*, Advances in differential geometry and topology, World Sci. Publ., Teaneck, NJ, 1990, pp. 1–38.

[14] M. Kac, *On the average number of real roots of a random algebraic equation*, Bull. Amer. Math. Soc. **49** (1943), 314–320.

[15] ———, *On the average number of real roots of a random algebraic equation. II*, Proc. London Math. Soc. (2) **50** (1949), 390–408.

[16] B. Ya. Kazarnovskiĭ, *Newton polyhedra and roots of systems of exponential sums*, Funktsional. Anal. i Prilozhen. **18** (1984), no. 4, 40–49, 96 (Russian).

[17] ———, *"Newton polyhedra" of generalized functions*, Izv. Ross. Akad. Nauk Ser. Mat. **68** (2004), no. 2, 53–70, DOI 10.1070/IM2004v068n02ABEH000475 (Russian, with Russian summary); English transl., Izv. Math. **68** (2004), no. 2, 273–289.

[18] Steven G. Krantz, *Function theory of several complex variables*, AMS Chelsea Publishing, Providence, RI, 2001. Reprint of the 1992 edition.

[19] A. G. Kušnirenko, *Newton polyhedra and Bezout's theorem*, Funkcional. Anal. i Priložen. **10** (1976), no. 3, 82–83. (Russian).

[20] J. E. Littlewood and A. C. Offord, *On the number of real roots of a random algebraic equation. III*, Rec. Math. [Mat. Sbornik] N.S. **12(54)** (1943), 277–286 (English, with Russian summary).

[21] ———, *On the distribution of the zeros and a-values of a random integral function. I*, J. London Math. Soc. **20** (1945), 130–136.

[22] Gregorio Malajovich, *Nonlinear Equations*, Publicações de Matemática, 28º Colóquio Brasileiro de Matemática, IMPA, Rio de Janeiro, 2011.

[23] Gregorio Malajovich and J. Maurice Rojas, *Polynomial systems and the momentum map*, Foundations of computational mathematics (Hong Kong, 2000), World Sci. Publ., River Edge, NJ, 2002, pp. 251–266.

[24] ———, *High probability analysis of the condition number of sparse polynomial systems*, Theoret. Comput. Sci. **315** (2004), no. 2-3, 524–555, DOI 10.1016/j.tcs.2004.01.006.

[25] Bernard Shiffman and Steve Zelditch, *Random polynomials with prescribed Newton polytope*, J. Amer. Math. Soc. **17** (2004), no. 1, 49–108 (electronic), DOI 10.1090/S0894-0347-03-00437-5.

[26] Michael Shub and Steve Smale, *Complexity of Bézout's theorem. I. Geometric aspects*, J. Amer. Math. Soc. **6** (1993), no. 2, 459–501, DOI 10.2307/2152805.

[27] M. Shub and S. Smale, *Complexity of Bezout's theorem. II. Volumes and probabilities*, Computational algebraic geometry (Nice, 1992), Progr. Math., vol. 109, Birkhäuser Boston, Boston, MA, 1993, pp. 267–285.

[28] Michael Shub and Steve Smale, *Complexity of Bezout's theorem. III. Condition number and packing*, J. Complexity **9** (1993), no. 1, 4–14, DOI 10.1006/jcom.1993.1002. Festschrift for Joseph F. Traub, Part I.

[29] ———, *Complexity of Bezout's theorem. IV. Probability of success; extensions*, SIAM J. Numer. Anal. **33** (1996), no. 1, 128–148, DOI 10.1137/0733008.

[30] M. Shub and S. Smale, *Complexity of Bezout's theorem. V. Polynomial time*, Theoret. Comput. Sci. **133** (1994), no. 1, 141–164, DOI 10.1016/0304-3975(94)90122-8. Selected papers of the Workshop on Continuous Algorithms and Complexity (Barcelona, 1993).

[31] Ferdinand Minding, *On the determination of the degree of an equation obtained by elimination*, Topics in algebraic geometry and geometric modeling, Contemp. Math., vol. 334, Amer. Math. Soc., Providence, RI, 2003, pp. 351–362. Translated from the German (Crelle, 1841)and with a commentary by D. Cox and J. M. Rojas.

[32] J. Maurice Rojas, *On the average number of real roots of certain random sparse polynomial systems*, The mathematics of numerical analysis (Park City, UT, 1995), Lectures in Appl. Math., vol. 32, Amer. Math. Soc., Providence, RI, 1996, pp. 689–699.

[33] Hermann Weyl, *The theory of groups and quantum mechanics*, Dover Publications, New York, 1949. XVII+422 pp.

Departamento de Matemática Aplicada, Instituto de Matemática da UFRJ. C.P.68530, Rio de Janeiro, RJ 21941-909, Brasil

*URL*: `http://www.labma.ufrj.br/~gregorio`

*E-mail address*: `gregorio.malajovich@gmail.com`